

Probabilistic counterparts of nonlinear parabolic PDE systems.

Ya.I. Belopolskaya,
St.Petersburg State University for Architecture and Civil Engineering,
St.Petersburg, Russia, yana@yb1569.spb.edu

Abstract

We extend the results of the FBSDE theory in order to construct a probabilistic representation of a viscosity solution to the Cauchy problem for a system of quasilinear parabolic equations. We derive a BSDE associated with a class of quasilinear parabolic system and prove the existence and uniqueness of its solution. To be able to deal with systems including nondiagonal first order terms along with the underlying diffusion process we consider its multiplicative operator functional. We essentially exploit as well the fact that the system under consideration can be reduced to a scalar equation in a enlarged phase space. This allows to obtain some comparison theorems and to prove that a solution to FBSDE gives rise to a viscosity solution of the original Cauchy problem for a system of quasilinear parabolic equations.

1 Introduction

Quasilinear systems of parabolic equations arise as mathematical models which describe various chemical and biological phenomena. They arise as well in financial mathematics and in differential geometry when one considers nonlinear parabolic equations in sections of vector bundles.

Let d, d_1 be given integers, $a(x) \in R^d, A(x) \in R^{d \times d}, B(x) \in R^{d \times d_1 \times d_1}, c(x) \in R^{d_1 \times d_1}, x \in R^d$ and $g : R^d \times R^{d_1} \times R^{d \times d_1} \rightarrow R_1^d$ be given. Consider a class of quasilinear parabolic equations of the form

$$\frac{\partial u_l}{\partial s} + \frac{1}{2} \text{Tr} A^* \nabla^2 u_l A + \langle a, \nabla u_l \rangle + \quad (1.1)$$

$$+ B_{lm}^i \nabla_i u_m + c_{lm} u_m + g_l(s, x, u, \nabla u) = 0, \quad u_l(T, x) = u_{0l}(x), \quad l = 1, \dots, d_1$$

with respect of R^{d_1} -valued function $u(s, x)$ defined on $[0, T] \times R^d$. Here and below we assume a convention of summing up over repeating indices if the contrary is not mentioned and denote by $\langle \cdot, \cdot \rangle$ an inner product in R^d regardless of d .

One can suggest at least a couple of probabilistic counterparts of the Cauchy problem (1.1). To derive them let us assume first that there exists a classical solution $u(s, x)$ to this problem. In this case one can prove applying the standard technique of the stochastic differential equation theory and especially the Ito formula, that the function $u(s, x)$ satisfying (1.1) admits at least two probabilistic representations.

The first one was suggested in papers by Dalecky and Belopolskaya [1] -[3] and was aimed to develop a probabilistic approach to prove the existence and uniqueness of a classical solution to (1.1) and as well as to much more general systems of the form

$$\frac{\partial u_l}{\partial s} + F(x, u, \nabla u, \nabla^2 u_l) = 0, \quad u_l(T, x) = u_{l0}(x).$$

The second one suggested in papers by Pardoux and Peng [4]- [6] leads to the powerful backward stochastic differential equations (BSDE) theory. This approach allows to construct a viscosity solution to a quasilinear scalar parabolic PDE or to a diagonal system of PDEs (see [6] - [7]). In terms of (1.1) this means that one have to set $B \equiv 0$ and $c \equiv 0$ and $g_l(x, u, A^* \nabla u) \equiv g_l(x, u, A^* \nabla u_l)$.

To present these approaches we fix a probability space (Ω, \mathcal{F}, P) and denote by $w(t) \in R^d$ the standard Wiener process. Let \mathcal{F}_t be a flow of σ -subalgebras of \mathcal{F} generated by $w(t)$ and $E_{s,x}[f(\xi(T))] = E[f(\xi(T)) | \xi(s) = x]$ denote the conditional expectation.

Assume that g in (1.1) does not depend on ∇u and all coefficients a, A, B, C depend on s, x and u . Assume that $u(s, x)$ is a smooth function satisfying (1.1) with these parameters. Then it was stated in [1] that this function admits a representation of the form

$$\langle h, u(s, x) \rangle = E_{s,x} \left[\langle \eta(T), u_0(\xi(T)) \rangle + \int_s^T \langle \eta(\theta), g(\theta, \xi(\theta), u(\theta, \xi(\theta))) d\theta \right], \quad (1.2)$$

where stochastic processes $\xi(t)$ and $\eta(t)$ satisfy the stochastic equations

$$d\xi(t) = a(\xi(t), u(t, \xi(t)))dt + A(\xi(t), u(t, \xi(t)))dw(t), \quad \xi(s) = x, \quad (1.3)$$

and

$$d\eta(t) = c(\xi(t), u(t, \xi(t)))\eta(t)dt + C(\xi(t), u(t, \xi(t)))\langle \eta(t), dw(t) \rangle, \quad \eta(s) = h. \quad (1.4)$$

Note that a, A, c in (1.3), (1.4) are the same as in (1.1) while it is assumed that C in (1.4) and B in (1.1) satisfy an equality $B_k^{lm} = C_i^{lm} A_{ik}$.

Remark. Notice that when A is a nondegenerated matrix one can define C by $C_i^{lm} = B_k^{lm} A_{ki}^{-1}$ while when A is a degenerated matrix we assume that B has the above form. It is important that although in a general case $A = 0$ yields $B = 0$ and we do not obtain a general nonlinear system of hyperbolic equations as a vanishing viscosity limit of (1.1). Nevertheless one can state some restrictions on B such that given $A_\epsilon = \epsilon A$ and $C_\epsilon = \epsilon^{-1} C$ one can apply (1.2) to investigate the vanishing viscosity limit of (1.1) with these coefficients (see AB) .

An important observation is the fact that we can consider (1.2)-(1.4) as a closed system of equations and state conditions on its data to ensure the existence and uniqueness of a solution to this system. If in addition it will be revealed that the function $u(s, x)$ given by (1.2) is twice differentiable in the spatial variable x , then one can verify that $u(s, x)$ is a unique classical solution of (1.1) with correspondent parameters. It should be mentioned that this approach can be essentially generated to give a possibility to study systems of quasilinear and even fully nonlinear parabolic equations. In other words one can consider (1.1) with coefficients a, A, c, C, g depending on $(x, u, \nabla u)$ or even $(x, u, \nabla u, \nabla^2 u)$. Note that to deal with these more complicated cases within a framework of this approach we require more strong assumptions concerning regularity of coefficients of (1.3)-(1.4) and the Cauchy data u_0 . As a result we can prove on this way the existence and uniqueness of a classical solution to (1.1), possibly on a small time interval.

To describe the second approach which allows to construct a different class of solutions to the Cauchy problem

$$\frac{\partial u_l}{\partial s} + \frac{1}{2} \text{Tr} A^*(x) \nabla^2 u_l A(x) + \langle a(x), \nabla u_l \rangle + g_l(x, u, A^* \nabla u_l) = 0, \quad u_l(T, x) = u_{0l}(x), \quad (1.5)$$

we assume once again that there exists a classical solution $u_l(s, x)$ of (1.5).

Consider a stochastic process $\xi(t)$ satisfying (1.3) with coefficients $a(s, x, u) \equiv a(s, x)$, $A(s, x, u) \equiv A(s, x)$. Keeping in mind that $u_l(s, x)$ is a classical solution of (1.5), by Ito's formula we derive an expression for a stochastic differential of $y(t) = \Gamma^*(s, t) u(t, \xi(t))$ in the form

$$dy(t) = -g(t, \xi(t), y(t), z(t))dt - zdw(t), \quad y(T) = \Gamma^*(s, T) u_0(\xi(T)), \quad (1.6)$$

where $z(t) = A^*(\xi(t)) \nabla u(t, \xi(t))$, $\eta(t) = \Gamma(s, t)h$. The equation (1.6) is called a backward stochastic equation (BSDE).

In general one can forget about the process $\xi(t)$ and consider an independent BSDE of the form

$$dy(t) = -f(t, y(t), z(t))dt - zdw(t), \quad y(T) = \zeta, \quad (1.7)$$

where $f(t, y, z)$ is an \mathcal{F}_t -adapted random process meeting some additional requirements and ζ is an \mathcal{F}_T -measurable random variable. A general theory of BSDEs was developed by a number of authors (see e.g. [7] for references). In addition the system (1.4), (1.6) shows a way to construct the so called viscosity solution to (1.5) (defined in [8]) setting $u(s, x) = y(s)$.

To generalize this approach and apply it to (1.1) we observe that this system has a crucial property which can be easily revealed if one analyzes the probabilistic representation (1.2) of a smooth solution to (1.1). Namely, the Cauchy problem (1.1) can be reduced to the Cauchy problem for a scalar equation

$$\frac{\partial \Phi}{\partial s} + \frac{1}{2} \text{Tr} Q^*(x, h) \nabla^2 \Phi Q(x, h) + \langle q(x, h), \nabla \Phi \rangle + G(s, h, x, \Phi, Q^* \nabla \Phi) = 0, \quad (1.8)$$

$$\Phi(T, x) = \Phi_0(x, h) = \langle h, u_0(x) \rangle.$$

with respect to a scalar function $\Phi(s, x, h) = \langle h, u(s, x) \rangle$.

Here

$$\begin{aligned} \text{Tr} Q^* \nabla^2 \Phi(s, x, h) Q &= A_{ki}^* \frac{\partial^2 \Phi(s, x, h)}{\partial x_i \partial x_j} A_{jk} + 2C_k^{lm} h_l \frac{\partial^2 \Phi(s, x, h)}{\partial x_j \partial h_m} A_{jk} + \\ &+ C_k^{qm} h_m \frac{\partial^2 \Phi(s, x, h)}{\partial h_q \partial h_p} C_k^{pn} h_n = A_{ki}^* \frac{\partial^2 \Phi(s, x, h)}{\partial x_i \partial x_j} A_{jk} + 2C_k^{lm} h_l \frac{\partial^2 \Phi(s, x, h)}{\partial x_j \partial h_m} A_{jk}, \end{aligned}$$

since, due to linearity of $\Phi(s, x, h)$ in h , we have $\frac{\partial^2 \Phi(s, x, h)}{\partial h_q \partial h_p} \equiv 0$. In addition

$$\langle q, \nabla \Phi(s, x, h) \rangle = a_j \frac{\partial \Phi(s, x, h)}{\partial x_j} + c_{lm} h_m \frac{\partial \Phi(s, x, h)}{\partial h_l}, \quad G(s, x, h) = \langle h, g(s, x, u, A^* \nabla u) \rangle.$$

Coming back to (1.4) we notice that its solution (provided it exists) gives rise to a multiplicative operator functional $\Gamma(t, s, \xi(\cdot)) \equiv \Gamma(t, s)$ of the process $\xi(t)$ satisfying (1.3), that is $\eta(t) = \Gamma(t, s)h$ and $\Gamma(t, s)h = \Gamma(t, \theta)\Gamma(\theta, s)$ a.s. for $0 \leq s \leq \theta \leq t \leq T$. Hence to derive an FBSDE associated with (1.1) we can proceed as follows.

Assume that there exists a classical solution to the Cauchy problem (1.1) or what is equivalent suppose that there exists a classical solution to (1.8) and compute a stochastic differential of a stochastic process $Y(t) = \langle \eta(t), u(t, \xi(t)) \rangle$,

$$dY(t) = \langle d\eta(t), u(t, \xi(t)) \rangle + \langle \eta(t), du(t, \xi(t)) \rangle + \langle d\eta(t), du(t, \xi(t)) \rangle.$$

Taking into account (1.3), (1.4) by Ito's formula we derive the relation

$$dY(t) = -F(t, Y(t), Z(t))dt + \langle Z(t), dW(t) \rangle, \quad Y(T) = \zeta = \langle \eta(T), u_0(\xi(T)) \rangle, \quad (1.9)$$

where $W(t) = (w(t), w(t))^*$,

$$\begin{aligned} \langle Z(t), dW(t) \rangle &= \langle C(\Gamma(t)h, dw(t)), u(t, \xi(t)) \rangle + \langle \Gamma(t)h, \nabla u(t, \xi(t))Adw \rangle = \\ &= \langle h, \Gamma^*(t)[C^*u(t, \xi(t)) + A^*\nabla u(t, \xi(t))]dw(t) \rangle \end{aligned}$$

and $\Gamma(t)h \equiv \Gamma(t, s)h = \eta_{s,h}(t)$. As a result we can rewrite (1.9) in the form

$$dy(t) = -f(t, y(t), z(t))dt + z(t)dw(t), \quad y(T) = \Gamma^*(s, T)u_0(\xi(T)), \quad (1.10)$$

where

$$\begin{aligned} f(t, y(t), z(t)) &= \\ &= \Gamma^*(t)g(\xi(t), u(t, \xi(t)), C^*(t, \xi(t))u(t, \xi(t)) + A^*(t, \xi(t))\nabla u(t, \xi(t))) = \\ &= \Gamma^*(t)g(\xi(t), [\Gamma^*]^{-1}(t)y(t), C^*(\xi(t))[\Gamma^*]^{-1}(t)y(t) + A^*(\xi(t))[\Gamma^*]^{-1}(t)z(t)), \\ Z(t) &= ([\Gamma^*]^{-1}(t)C^*(t, \xi(t))u(t, \xi(t)), [\Gamma^*]^{-1}(t)A^*(\xi(t))\nabla u(t, \xi(t)))^*, \\ z(t)dw(t) &= [\Gamma^*]^{-1}(t)[C^*udw(t) + A^*\nabla udw(t)] \in R^{d_1} \end{aligned} \quad (1.11)$$

and $\langle h, z(t)dw(t) \rangle = \langle Z(t), dW(t) \rangle$.

When the solution $y(t)$ is a scalar process and a comparison theorem holds one can prove that the function $u(s, x)$ defined by $y(s) = u(s, x)$ is a viscosity solution of the Cauchy problem for a corresponding quasilinear parabolic equation. In a multidimensional case it was shown in [9] that given a solution of the BSDE

$$dy_l(t) = -g_l(t, \xi(t), y(t), z_l(t))dt + \langle z_l(t), dw(t) \rangle, \quad y(T) = \Gamma^*(s, T)u_0(\xi(T)), \quad (1.12)$$

where $\xi(t)$ satisfies (1.9) under some condition one can prove that the function $u(s, x) = y(s)$ is a viscosity solution to the Cauchy problem

$$\frac{\partial u_l}{\partial s} + \frac{1}{2}Tr A^* \nabla^2 u_l A + \langle a, \nabla u_l \rangle + g_l(s, x, u, A^* \nabla u_l) = 0, \quad u_l(T, x) = u_{0l}(x). \quad (1.13)$$

In this paper we show that a certain combination of two approaches allows to extend the results of forward-backward stochastic equations (FBSDEs) theory to construct a viscosity solution to the system of the form (1.1). In particular we define the very notion of a viscosity solution for (1.1) and prove a comparison theorem for solutions of multidimensional BSDEs which is a crucial point in construction of the viscosity solution via a solution to a BSDE.

In the next section we give a construction of an FBSDE required to construct a viscosity solution for (1.1), assuming that coefficients a, σ, C, c do not depend on u . We state here conditions on the BSDE parameters that ensure the existence and uniqueness of its solution. In section 3 we prove a comparison theorem and in section 4 we state the notion of a viscosity solution of the Cauchy problem for (1.1) and prove that FBSDE solution gives rise to a viscosity solution for (1.1).

2 Forward-backward stochastic differential equations

In this section we introduce notations and present in a suitable form necessary results from FBSDE theory adapted to the case under consideration.

Given integers d, d_1 consider Euclidian spaces R^d, R^{d_1} and let $\|\cdot\|$ denote a norm in R^d and $\langle \cdot, \cdot \rangle$ denote an inner product regardless of d .

Given a Euclidian space X let

- $L_t^p(X)$ be a set of \mathcal{F}_t -measurable X -valued random variables, $E\|\xi\|^p < \infty$;
- $\mathcal{H}_c^2(X)$ be a set of \mathcal{F}_t -measurable X -valued semimartingales such that $E\left[\sup_{0 \leq t \leq T} \|y(t)\|^2\right] < \infty$;
- $\mathcal{H}_t^2(X)$ be a set of $\mathcal{F}_{s,t}$ -measurable X -valued semimartingales such that $E\left[\sup_{0 \leq \theta \leq t} \|y(\theta)\|^2\right] < \infty$;
- $\mathcal{H}^2(X)$ be a set of square integrable progressively measurable processes $z(t) \in X$ such that $E\left[\int_0^T \|z(\tau)\|^2 d\tau\right] < \infty$;
- $\mathcal{S}^2 = \mathcal{H}_c^2(R^{d_1}) \cup \mathcal{H}^2(R^{d \times d_1})$;
- $\mathcal{S}^3 = \mathcal{H}_c^2(R^d) \cup \mathcal{H}_c^2(R^{d_1}) \cup \mathcal{H}_T^2(R^{d \times d_1})$;
- $\mathcal{B}^2 = \mathcal{H}^2(R^{d_1}) \cup \mathcal{H}_T^2(R^{d \times d_1})$;
- $\mathcal{B}^3 = \mathcal{H}^2(R^d) \cup \mathcal{H}^2(R^{d_1}) \cup \mathcal{H}_T^2(R^{d \times d_1})$;
- $L(R^d)$ be the space of bounded linear maps acting in R^d ;
- $L(R^d; R^{d_1}) \equiv R^{d \times d_1}$ be the space of bounded linear maps acting from R^d to R^{d_1} ;

• Given $\beta > 0$ and $\phi \in \mathcal{H}_T^2(R^d)$ let $\|\phi\|_\beta^2 = E \left[\int_0^T e^{\beta t} \|\phi(t)\|^2 dt \right]$ and $\mathcal{H}_{T,\beta}^2(R^d)$ be the space $\mathcal{H}_T^{2,d}$ equipped with the norm $\|\cdot\|_\beta$.

Let $W(t) = (w(t), \eta(t)) \in R^d \times R^{d_1}$ and $\kappa(t) = (\xi(t), \eta(t)) \in R^d \times R^{d_1}$ be a solution of a system of SDEs

$$d\xi(t) = a(t, \xi(t))dt + A(t, \xi(t))dw(t), \quad \xi(s) = x \in R^d, \quad (2.1)$$

$$d\eta(t) = c(t, \xi(t))\eta(t)dt + C(t, \xi(t))(\eta(t), dw(t)), \quad \eta(s) = h \in R^{d_1}. \quad (2.2)$$

We say that condition **C 2.1** holds if coefficients $a : [0, \infty) \times R^d \rightarrow R^d$, $A : [0, \infty) \times R^d \rightarrow L(R^d)$, $c : [0, \infty) \times R^d \rightarrow L(R^{d_1})$, $C : [0, \infty) \times R^d \rightarrow L(R^d; L(R^{d_1}))$ are continuous in $t \in [0, T]$ and there exist constants K_1, K_2, L_1, L_2 such that

$$\|a(t, x)\|^2 + \|A(t, x)\|^2 \leq K_1[1 + \|x\|^2];$$

$$\|a(t, x_1) - a(t, x_2)\|^2 + \|A(t, x_1) - A(t, x_2)\|^2 \leq L_1\|x_1 - x_2\|^2;$$

$$\|c(t, x)h\|^2 + \|C(t, x)h\|^2 \leq K_2\|h\|^2;$$

$$\|c(t, x_1) - c(t, x_2)h\|^2 + \|[C(t, x_1) - C(t, x_2)]h\|^2 \leq L_2\|x_1 - x_2\|^2\|h\|^2.$$

Recall that we use notation $\|A\| = [\sum_{j,k=1}^d A_{kj}A_{jk}]^{\frac{1}{2}}$ for $A \in L(R^d)$.

Lemma 2.1. Let condition **C 2.1** hold. Then there exists a unique solution $\kappa(t) = (\xi(t), \eta(t)) \in R^d \times R^{d_1}$ to (2.1), (2.2) such that $\xi(t) \in R^d$ is a Markov process with $E\|\xi(t)\|^2 < \infty$ and $\eta(t) \in R^{d_1}$ with $E\|\eta(t)\|^2 < \infty$ for any $t \in [0, T]$.

It follows from **C 2.1** that coefficients of equations (2.1) and (2.2) satisfy classical conditions of the existence and uniqueness theorem for solutions of SDEs and hence the lemma statement results from this theorem.

Lemma 2.2. Let condition **C 2.1** hold. Then the stochastic process $\eta(t)$ satisfying (2.2) gives rise to a multiplicative operator functional $\Gamma(t) \equiv \Gamma(t, s) : \mathcal{H}_s^2(R^{d_1}) \rightarrow \mathcal{H}_t^2(R^{d_1})$ satisfying the SDE

$$d\Gamma(t) = c(t, \xi(t))\Gamma(t)dt + C(t, \xi(t))(\Gamma(t), dw(t)), \quad \Gamma(s, s) = I, \quad (2.3)$$

where I is the identity operator in R^{d_1} . Moreover there exists an inverse map $\Gamma^{-1}(s, t) : \mathcal{H}_t^2(R^{d_1}) \rightarrow \mathcal{H}_s^2(R^{d_1})$ satisfying

$$\Gamma^{-1}(s, t) = I - \int_s^t \Gamma^{-1}(\theta, t)[c(\theta, \xi(\theta)) - C^2(\theta, \xi(\theta))]d\theta - \int_s^t \Gamma^{-1}(t)C(\theta, \xi(\theta))dw(\theta) \quad (2.4)$$

with probability 1.

Proof. Under the condition **C 2.1** we can state the existence and uniqueness of a solution to (2.4) and the corresponding properties of the map $\Gamma^{-1}(s, t)$. In particular we deduce from uniqueness of solutions to (2.2) and (2.4) that the map $\Gamma(t, s)$ defined by $\eta(t) = \Gamma(t, s)h$ is an evolution family, that is $\Gamma(t, \theta)\Gamma(\theta, s) = \Gamma(t, s)$ with probability 1 and the map $\Gamma^{-1}(t, s)$ has the same property. Besides by Ito's formula we can check that $\Gamma(t, s)\Gamma^{-1}(s, t) = I$ a.s.

Let $\Gamma^*(s, t)$ be defined by $\langle \Gamma(t, s)h, u \rangle = \langle h, \Gamma^*(s, t)u \rangle$. We can verify that $\Gamma^*(s, t)$ is an invertible evolution map acting from $\mathcal{H}_t^2(R^{d_1})$ to $\mathcal{H}_s^2(R^{d_1})$. Here and below we identify the space R^d with its dual space $(R^d)^*$.

Consider a BSDE of the form

$$dy(t) = -\Gamma^*(s, t)g([\Gamma^*]^{-1}(s, t)y(t), [\Gamma^*]^{-1}(s, t)z(t))dt + z(t)dw(t), \quad y(T) = \zeta, \quad (2.5)$$

and state conditions on its parameters g and ζ to ensure that there exists a unique solution $(y(t) \in R_1^d, z(t) \in R^{d \times d_1})$ to (2.5).

We say that condition **C 2.2** holds when:

$g : [s, T] \times R^d \times R^{d_1} \times R^{d \times d_1} \rightarrow R^{d_1}$, $\zeta \in R^{d_1}$ be an \mathcal{F}_T -measurable square integrable random variable and there exist constants L, L_3 , such that

$$\|g(t, x^1, y, z) - g(t, x^2, y, z)\| \leq L_3\|x^1 - x^2\|,$$

$$\|g(t, x, y^1, z^1) - g(t, x, y^2, z^2)\| \leq L[\|y^1 - y^2\| + \|z^1 - z^2\|],$$

$$\langle y - y_1, g(t, x, y^1, z) - g(t, x, y^2, z) \rangle \leq \mu\|y - y^1\|^2,$$

3) There exists a constant $C_0 > 0$ such that for all $x, x' \in R^d$

$$\|u_0(x) - u_0(x')\| \leq C_0\|x - x'\|.$$

Denote by $f(t, y, z) = \Gamma^*(t)g(\xi(t), [\Gamma^*]^{-1}(t)y, [\Gamma^*]^{-1}(t)z)$ and let $\zeta = \Gamma^*(s, T)u_0(\xi(T))$, where $\xi(t), t \in [s, T]$ is a solution to (2.1). Consider a BSDE

$$dy(t) = -f(t, \xi(t), y(t), z(t))dt + z(t)dw(t), \quad y(T) = \zeta \in R^{d_1}. \quad (2.6)$$

A couple of progressively measurable random processes $(y(t), z(t)) \in \mathcal{B}^2$ is called a solution of (2.6) if with probability 1

$$y(t) = \zeta + \int_t^T f(\theta, \xi(\theta), y(\theta), z(\theta)) ds - \int_t^T z(\theta) dw(\theta), \quad 0 \leq t \leq T. \quad (2.7)$$

Lemma 2.3. Let conditions **C 2.1**, **C 2.2** hold. Then

$$\|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)\| \leq L[\|y_1 - y_2\| + \|z_1 - z_2\|].$$

Proof. By Lipschitz continuity of g and the properties of $\Gamma(t)$ we have a.s.

$$\begin{aligned} \|f(t, y_1, z_1) - f(t, y_2, z_2)\| &= \|g(t, \xi(t), [\Gamma^*]^{-1}y_1, [\Gamma^*]^{-1}z_1) - g(t, \xi(t), [\Gamma^*]^{-1}y_2, [\Gamma^*]^{-1}z_2)\| \\ &\leq \|\Gamma^*\|L[\|[\Gamma^*]^{-1}y_1 - [\Gamma^*]^{-1}y_2\| + \|[\Gamma^*]^{-1}z_1 - [\Gamma^*]^{-1}z_2\|] \leq L[\|y_1 - y_2\| + \|z_1 - z_2\|]. \end{aligned}$$

Given $(u, v) \in \mathcal{B}^2$, we define a map M by $(y, z) = M(u, v)$ as follows. Let ζ be R^{d_1} -valued \mathcal{F}_T -measurable random variable and given $f : [s, T] \times R^d \times R^{d_1} \times R^{d \times d_1} \rightarrow R^{d_1}$ set

$$y(t) = E[\zeta + \int_t^T f(\theta, \xi(\theta), u(\theta), v(\theta)) d\theta | \mathcal{F}_t], \quad 0 \leq t \leq T. \quad (2.8)$$

We apply the Ito theorem about martingale representation of a square integrable random variable

$$\chi = \zeta + \int_0^T f(\theta, u(\theta), v(\theta)) d\theta$$

to define the process $z(t)$ by the equality

$$\chi = E[\chi] + \int_0^T z(\theta) dw(\theta).$$

It is easy to check that the couple (y, z) defined in this way satisfies

$$y(t) = \zeta + \int_t^T f(\theta, \xi(\theta), u(\theta), v(\theta)) d\theta - \int_t^T z(\theta) dw(\theta).$$

In a standard way we show that M acts in \mathcal{B}^2 and possesses a contraction property. To this end we denote by $\bar{f} = f_1 - f_2$ for $f = y, z, u, v$. By Ito's formula we obtain

$$\begin{aligned} &e^{\beta t} E\|\bar{y}(t)\|^2 + E\left[\int_t^T e^{\beta s} [\beta \|\bar{y}(s)\|^2 + \|\bar{z}(s)\|^2] ds\right] = \\ &= 2E\left[\int_t^T e^{\beta s} \langle \bar{y}(s), f(s, u_1(s), v_1(s)) - f(s, u_2(s), v_2(s)) \rangle ds\right]. \end{aligned}$$

Taking into account Lipschitz continuity of f we obtain

$$\begin{aligned} &E[e^{\beta t} \|\bar{y}(t)\|^2] + E\left[\int_t^T e^{\beta s} [\beta \|\bar{y}(s)\|^2 + \|\bar{z}(s)\|^2] ds\right] \leq \\ &\leq 2LE\left[\int_t^T e^{\beta s} \|\bar{y}(s)\| [\|\bar{u}(s)\| + \|\bar{v}(s)\|] ds\right] \end{aligned}$$

and by the elementary inequality $2ab \leq a^2\alpha^2 + \frac{b^2}{\alpha^2}$,

$$\begin{aligned} &E[e^{\beta t} \|\bar{y}(t)\|^2] + E\left[\int_t^T e^{\beta s} \|\bar{z}(s)\|^2 ds\right] \leq \\ &\leq [2L^2\alpha^2 - \beta]E\left[\int_t^T e^{\beta s} \|\bar{y}(s)\|^2 ds\right] + \frac{1}{\alpha^2}E\left[\int_t^T e^{\beta s} (\|\bar{u}(s)\|^2 + \|\bar{v}(s)\|^2) ds\right]. \end{aligned}$$

Choosing $\frac{1}{\alpha^2} = \frac{1}{2}$ and $\beta - 4L^2 = 1$ we obtain

$$e^{\beta t} E\|\bar{y}(t)\|^2 + E\left[\int_t^T e^{\beta s} \|\bar{z}(s)\|^2 ds\right] \leq \frac{1}{2}E\left[\int_t^T e^{\beta s} [\|\bar{u}(s)\|^2 + \|\bar{v}(s)\|^2] ds\right].$$

In the similar way we can check that $(y, z) = M(u, v) \in \mathcal{B}^2$. As a result we deduce that M is a contraction in \mathcal{B}^2 and the following statement holds.

Theorem 2.1 Let condition **C 2.2** hold. Then there exists a unique solution $(y, z) \in \mathcal{B}^2$ of BSDE (2.6) and successive approximations (y^n, z^n) of the form

$$y^{n+1}(t) = \zeta + \int_t^T f(\theta, \xi(\theta), y^n(\theta), z^n(\theta))d\theta - \int_t^T z^{n+1}(\theta)dw(\theta)$$

converges to the solution of (2.6) with probability 1.

Proof. The existence and uniqueness of a solution (y, z) to (2.6) follows from the fixed point theorem for the contraction $M : \mathcal{B}^2 \rightarrow \mathcal{B}^2$. Applying the above estimates to the successive approximations (y^n, z^n) we can verify that

$$E \left[\int_t^T e^{\beta\theta} \|y^n(\theta) - y^m(\theta)\|^2 ds | \mathcal{F}_t \right] + E \left[\int_t^T e^{\beta\theta} \|z^n(\theta) - z^m(\theta)\|^2 ds | \mathcal{F}_t \right] \rightarrow 0, \quad m, n \rightarrow \infty$$

with probability 1. Hence, (y^n, z^n) is a Cauchy sequence in \mathcal{B}^2 and the limit $P - \lim_{n \rightarrow \infty} (y^n, z^n) = (y, z)$ exists and satisfies (2.4).

Below along with a weakly coupled multidimensional FBSDE of the form

$$dy(t) = -f(t, \xi(t), y(t), z(t))dt + z(t)dw(t), \quad y(T) = \Gamma^*(s, T)u_0(\xi(T)), \quad (2.9)$$

where $\xi(t)$ is a solution of (2.4) we consider a weakly coupled scalar FBSDE which can be described as follows. Let

$$q(\kappa) = \begin{pmatrix} a(x) \\ c(x)h \end{pmatrix}, Q(\kappa) = \begin{pmatrix} A(x) & 0 \\ 0 & C(x)h \end{pmatrix}, \quad \tilde{G}(\kappa, y, z) = \langle h, f(x, y, z) \rangle. \quad (2.10)$$

Obviously, we can rewrite the system (2.1),(2.2) in the form

$$d\kappa(t) = q(t, \kappa(t))dt + Q(t, \kappa(t))dW(t), \quad \kappa(s) = \kappa = (x, h), \quad (2.11)$$

The required FBSDE can be presented in the form

$$dY(t) = -\tilde{G}(t, \kappa(t), Y(t), Z(t))dt + \langle Z(t), dW(t) \rangle, \quad Y(T) = \langle \eta(T), u_0(\xi(T)) \rangle, \quad (2.12)$$

where $\kappa(t) = (\xi(t), \eta(t))$ solves (2.11), $W(t) = (w(t), w(t))^*$ and $\langle Z(t), dW(t) \rangle = \langle h, z(t)dw(t) \rangle$.

A triple of progressively measurable random processes $(\kappa(t), y(t), z(t)) \in \mathcal{B}^3$ is called a solution of (2.11),(2.12) if with probability 1 for all $0 \leq s \leq t \leq T$

$$\kappa(t) = \kappa + \int_s^t q(\theta, \kappa(\theta))d\theta + \int_s^t Q(\theta, \kappa(\theta))dW(\theta), \quad (2.13)$$

$$Y(t) = \langle \eta(T), u_0(\xi(T)) \rangle + \int_t^T \tilde{G}(\theta, \kappa(\theta), Y(\theta), Z(\theta))d\theta - \int_t^T \langle Z(\theta), dW(\theta) \rangle. \quad (2.14)$$

The FBSDEs (2.1), (2.2), (2.6) and (2.11), (2.12) are equivalent.

3 Comparison theorem for multidimensional BSDE

Comparison theorems present an important tool in the BSDE and FBSDE theory and in particular in the context of the connections between FBSDE theory and viscosity solutions of corresponding parabolic equations and systems. In this paper to prove a comparison theorem for a multidimensional BSDE we use the special features of the BSDE under consideration.

Consider a couple of d_1 -dimensional BSDEs

$$y^i(t) = \zeta^i + \int_t^T f^i(\theta, y^i(\theta), z^i(\theta))d\theta - \int_t^T z^i(\theta)dw(\theta), \quad i = 1, 2 \quad (3.1)$$

for $0 \leq t \leq T$ and use the specific features of these BSDEs investigated in the previous sections. Here $\zeta^i, f^i(\theta, y, z) \in R^{d_1}$ for $\theta \in [0, T], y \in R^{d_1}, z \in R^{d \times d_1}$.

For any fixed nonzero vector $h \in R^{d_1}$ and $y^1, y^2 \in R^{d_1}$ we say that $y^1 \leq_h y^2$ under h if $\langle h, y^1 \rangle \leq \langle h, y^2 \rangle$. Without loss of generality we choose h to have $\|h\| = 1$.

Given two vectors $y^1, y^2 \in R^{d_1}$, we say $y^1 \leq y^2$ if $y_m^1 \leq y_m^2, m = 1, \dots, d_1$, where $y_m = \langle y, e_m \rangle$ and $(e_m)_{m=1}^{d_1}$ is a fixed orthonormal basis in R^{d_1} .

Given $f \in R^{d_1}$ we denote by $f_m^+ = \max[f_m, 0], m = 1, \dots, d_1$.

Consider a couple of BSDEs with parameters $\zeta^i, f^i, i = 1, 2$.

We say that condition **C 3.1** holds if

- i) $\zeta^1 \leq \zeta^2, P - \text{a.s.}$,
- ii) for each $m = 1, \dots, d_1$ inequality $f_m^1(t, y^1, z^1) \leq f_m^2(t, y^2, z^2)$ holds true when $y_l^1 \leq y_l^2$ for all $l = 1, \dots, d_1$ except $l = m$ while $y_m^1 = y_m^2$, and $z_{mk}^1 = z_{mk}^2$ for each $k = 1, \dots, d$,

iii) For all $y^1, y^2 \in R^{d_1}, z^1, z^2 \in R^{d \times d_1}$ and for each $m = 1, \dots, d_1$

$$\|f_m^i(t, y^1, z^1) - f_m^i(t, y^2, z^2)\| \leq L[\|y^1 - y^2\| + \|z^1 - z^2\|], i = 1, 2.$$

Set $\bar{\alpha} = \alpha^1 - \alpha^2$ for $\alpha = y, \zeta, f$ and z as well.

Let us mention that within this section we do not assume summing up with respect to repeating indices.

Theorem 3.1. Let (ζ^i, f^i) , $i = 1, 2$ be parameters of BSDEs (3.1) satisfying conditions **C 2.1** and **C 3.1**. Assume that $(y^i(t), z^i(t))$, $i = 1, 2$, $t \in [s, T]$ solve (3.1) with this parameters. Then $y^1(t) \leq y^2(t)$ a.s. Moreover the comparison is strict, that is if in addition $y^2(s) = y^1(s)$ then $\zeta^1 = \zeta^2$, $f^2(t, y^2(t), z^2(t)) = f^1(t, y^2(t), z^2(t))$ and $y^2(t) = y^1(t)$, $\forall t \in [s, T]$ P -a.s. In particular whenever either $P(\zeta^1 < \zeta^2) > 0$ or $f^1(t, y^2(t), z^2(t)) < f^2(t, y^2(t), z^2(t))$ on a set of positive $dt \times dP$ measure, then $y^1(s) < y^2(s)$ a.s.

Proof. Applying Ito's formula to $|\bar{y}_j(t)^+|^2$ where $j = 1, \dots, d_1$, and evaluating mean value we get

$$E|\bar{y}_j(t)^+|^2 = E|\bar{\zeta}_j^+|^2 - E\left[\int_t^T 2I_{y_j^1(s) > y_j^2(s)} \bar{y}_j(s) [f_j(s, y^1(s), z^1(s)) - f_j(s, y^2(s), z^2(s))] ds\right] - E\left[\int_t^T \bar{y}_j^+ dL_j(s)\right], \quad (3.2)$$

$$f_j(s, y^2(s), z^2(s))] ds - E\left[\int_t^T I_{\{y_j^1(s) > y_j^2(s)\}} \|\bar{z}_j(s)\|^2 ds\right] - E\left[\int_t^T \bar{y}_j^+ dL_j(s)\right],$$

where $L_j(t)$ is the local time of $\bar{y}_j(s)$ at 0. Note that the last summand is equal to 0 and since $\zeta^1 \leq \zeta^2$ a.s. we have $E[\|\zeta^1 - \zeta^2\|^2] = 0$. Obviously,

$$E\left[\int_t^T I_{y_j^1(s) > y_j^2(s)} \bar{y}_j(s) \bar{z}_j(s) dw(s)\right] = 0. \text{ Hence,}$$

$$E[\bar{y}_j(t)^+] = E\left[\int_t^T I_{y_j^1(s) > y_j^2(s)} 2\bar{y}_j^+(s) [f_j^1(s, y^1(s), z^1(s)) - f_j^2(s, y^2(s), z^2(s))] ds\right] - E\left[\int_t^T I_{\{y_j^1(s) > y_j^2(s)\}} \|\bar{z}_j(s)\|^2 ds\right].$$

Set

$$\begin{aligned} \bar{f}_j(s) &= f_j^1(s, y^1, z^1) - f_j^2(s, y^2, z^2) = \\ &= f_j^1(s, y_1^1, \dots, y_j^1, \dots, y_{d_1}^1, z_1^1, \dots, z_j^1, \dots, z_{d_1}^1) - \\ &= f_j^2(s, y_1^2, \dots, y_j^2, \dots, y_{d_1}^2, z_1^2, \dots, z_j^2, \dots, z_{d_1}^2) = \\ &= [f_j^1(s, y_1^1, \dots, y_j^1, \dots, y_{d_1}^1, z_1^1, \dots, z_j^1, \dots, z_{d_1}^1) - \\ &= f_j^2(s, y_1^1 + \bar{y}_1^+, \dots, y_j^1 + \bar{y}_j^+, \dots, y_{d_1}^1 + \bar{y}_{d_1}^+, z_1^1, \dots, z_j^1, \dots, z_{d_1}^1)] + \\ &+ [f_j^2(s, y_1^2 + \bar{y}_1^+, \dots, y_j^2 + \bar{y}_j^+, \dots, y_{d_1}^2 + \bar{y}_{d_1}^+, z_1^2, \dots, z_j^2, \dots, z_{d_1}^2) - \\ &= f_j^2(s, y_1^2, \dots, y_j^2, \dots, y_{d_1}^2, z_1^2, \dots, z_j^2, \dots, z_{d_1}^2)] = \Pi_1 + \Pi_2 \end{aligned}$$

Since for any $m = 1, \dots, d_1$ we have $y_m^1 \leq y_m^2 + \bar{y}_m^+$ for $m \neq j$, taking into account ii) in **C 3.1** we get $\Pi_1 \leq 0$.

Next, due to Lipschitz continuity of f^2 we have

$$\Pi_2 \leq L[|\bar{y}_1^+| + \dots + |\bar{y}_{j-1}^+| + |\bar{y}_j| + \dots + |\bar{y}_{d_1}^+| + \|\bar{z}_j\|].$$

Applying Ito's formula due to generator properties we deduce that

$$\begin{aligned} E|\bar{y}_j^+(t)|^2 &\leq 2E\left[\int_t^T I_{y_j^1(s) > y_j^2(s)} \bar{y}_j^+(s) \bar{f}_j(s) ds\right] - E\left[\int_t^T I_{y_j^1(s) > y_j^2(s)} \sum_{k=1}^d |\bar{z}_{jk}(s)|^2 ds\right] \leq \\ &\leq E\left[2\int_t^T I_{y_j^1(s) > y_j^2(s)} L\bar{y}_j^+(s) [|\bar{y}_1(s)| + \dots + |\bar{y}_{j-1}^+| + |\bar{y}_j(s)| + \dots + |\bar{y}_{d_1}^+| + \|\bar{z}_j(s)\|] ds\right] - \\ &- E\left[\int_t^T I_{\{y_j^1(s) > y_j^2(s)\}} \|\bar{z}_j(s)\|^2 ds\right] \leq E\left[\int_t^T I_{\{y_j^1(s) > y_j^2(s)\}} L^2(d_1 + 1) |\bar{y}_j(s)|^2 ds\right] + \\ &+ E\left[\int_t^T I_{\{y_j^1(s) > y_j^2(s)\}} \left[\sum_{k=1}^{d_1} |\bar{y}_k(s)|^2 + \|\bar{z}_j(s)\|^2\right] ds\right] - \\ &E\left[\int_t^T I_{\{y_j^1(s) > y_j^2(s)\}} \|\bar{z}_j(s)\|^2 ds\right] = L^2(d_1 + 1) \int_t^T E[I_{\{y_j^1(s) > y_j^2(s)\}} |\bar{y}_j(s)|^2] ds \\ &+ \int_t^T E[I_{\{y_j^1(s) > y_j^2(s)\}} \sum_{k=1}^{d_1} |\bar{y}_k(s)|^2] ds. \end{aligned} \quad (3.3)$$

Note that above we have used an elementary inequality of the form

$$2L\bar{y}_j^+(s)|\bar{y}_k(s)| \leq L^2|\bar{y}_j^+(s)|^2 + |\bar{y}_k(s)|^2.$$

Summing up left and right hand side in (3.3) we get that the function $m(t) = \sum_{j=1}^{d_1} E|\bar{y}_j^+(t)|^2$ satisfies inequality

$$m(t) \leq (L^2(d_1 + 1) + d_1) \int_t^T m(s) ds$$

Finally, due to results of the previous section we know that for $t \in [0, T]$ the inequality $E|\bar{y}_j(t)^+|^2 < \infty$ holds for each $j = 1, \dots, m$ then by the Gronwall lemma we know that $m(t) = 0$ and since m is a sum of positive summands, each summand should be equal to zero. Hence $|\bar{y}_j^+(t)| = 0$ and thus $y_j^1(t) \leq y_j^2(t)$ a.s. for all $j = 1, \dots, d_1$.

At the end of this section we come back to the one-dimensional BSDE (2.14) and derive the corresponding comparison theorem. Note that this theorem motivates our choice of comparison for vector functions in the case under consideration.

Consider the SDE of the form

$$\kappa(t) = \kappa + \int_s^t q(\kappa(\theta)) d\theta + \int_s^t Q(\kappa(\theta)) dW(\theta), \quad s \leq t \leq T, \quad (3.4)$$

introduced in the previous section and note that one can consider instead of the BSDE

$$y(t) = \Gamma^*(s, T)u_0(\xi(T)) + \int_t^T f(\theta, \xi(\theta), y(\theta), z(\theta)) d\theta - \int_t^T z(\theta) dw(\theta), \quad s \leq t \leq T, \quad (3.5)$$

with respect to the process $y(t) \in R^{d_1}$ a new BSDE

$$dY(t) = -\tilde{G}(t, \kappa(t), Y(t), Z(t)) dt + \langle Z(t), dW(t)(t) \rangle, \quad Y(T) = \zeta = \langle \eta(T), u_0(\xi(T)) \rangle, \quad (3.6)$$

where $Y(t) = \langle \eta(t), u(t, \xi(t)) \rangle$ is a scalar process. We denote $|Y| = \sup_{\|h\|=1} |\langle h, u \rangle| = \|u\|$.

Theorem 3.2. Let $(Y^i, Z^i), i = 1, 2$ be solutions of one dimensional BSDEs

$$dY^i(t) = -\tilde{G}^i(t, \kappa(t), Y^i(t), Z^i(t)) dt + \langle Z^i(t), dW(t) \rangle, \quad Y^i(T) = \Upsilon^i = \langle \eta(T), u_0^i(\xi(T)) \rangle. \quad (3.7)$$

Suppose that $\Upsilon^1 \leq \Upsilon^2$ and $\tilde{G}^1(t, \kappa, Y^2, Z^2) \leq \tilde{G}^2(t, \kappa, Y^2, Z^2)$ $dt \times dP$ - a.e. Then $Y^1(t) \leq Y^2(t)$ a.s. for all $s \leq t \leq T$.

Proof Define a scalar process

$$\mu(t) = \begin{cases} \frac{\tilde{G}^1(t, \kappa(t), Y^2(t), Z^1(t)) - \tilde{G}^1(t, \kappa(t), Y^1(t), Z^1(t))}{Y^2(t) - Y^1(t)} & \text{if } Y^1(t) \neq Y^2(t), \\ 0 & \text{if } Y^1(t) = Y^2(t), \end{cases}$$

and a vector process $\nu(t) \in R^d$ such that

$$\nu_k(t) = \begin{cases} \frac{\tilde{G}^1(t, \kappa(t), Y^1(t), Z^{(k)}(t)) - \tilde{G}^1(t, \kappa(t), Y^1(t), Z^{(k-1)}(t))}{Z_k^2(t) - Z_k^1(t)} & \text{if } Z_k^1(t) \neq Z_k^2(t) \\ 0 & \text{if } Z_k^1(t) = Z_k^2(t) \end{cases},$$

where $Z^{(k)}(t)$ denotes the d -dimensional vector such that its first k components are equal to corresponding components of Z^2 and the remaining $d - k$ components are equal to those of Z^1 . Due to Lipschitz continuity of g the processes $\mu(t)$ and $\nu(t)$ are bounded and in addition they are progressively measurable.

As above we use notation $\bar{f} = f^1 - f^2$ for $f = Y, Z, \Upsilon$ and observe that $(\bar{Y}(t), \bar{Z}(t))$ satisfies the BSDE

$$\bar{Y}(t) = \bar{\Upsilon} + \int_t^T [\mu(\theta)\bar{Y}(\theta) + \langle \nu(\theta), \bar{Z}(\theta) \rangle] d\theta + \int_t^T N(\theta) d\theta - \int_t^T \langle \bar{Z}(\theta), dW(\theta) \rangle,$$

where $N(t) = \tilde{G}^1(t, \kappa(t), Y^2(t), Z^2(t)) - \tilde{G}^2(t, \kappa(t), Y^2(t), Z^2(t))$. For $s \leq t \leq T$ we define

$$\rho_{s,t} = \exp \left[\int_s^t (\mu(\theta) - \frac{1}{2} \|\nu(\theta)\|^2) d\theta + \int_s^t \langle \nu(\theta), dW(\theta) \rangle \right].$$

By Ito's formula we can verify that $(\bar{Y}(\theta), \bar{Z}(\theta))$ satisfy the BSDE

$$d[\rho_{s,\theta} \bar{Y}(\theta)] = \rho_{s,\theta} [\bar{Y}(\theta) + N(\theta)] d\theta + \rho_{s,\theta} \langle \bar{Z}(\theta) + \bar{Y}(\theta) \nu(\theta), dW(\theta) \rangle$$

for $\theta \in [s, T]$ and

$$\bar{Y}(\theta) = E \left[\rho_{s,T} \bar{\Upsilon} + \int_\theta^T \rho_{s,\vartheta} N(\vartheta) d\vartheta | \mathcal{F}_\theta \right]$$

The required assertion immediately follows from negativity of $\tilde{\Upsilon}$ and $N(t)$.

Let us mention a useful remark. Let Y^1, Z^1 be a solution of BSDE

$$Y^1(t) = \Upsilon^1 + \int_t^T \tilde{G}^1(\theta, Y^1(\theta), Z^1(\theta))d\theta - \int_t^T \langle Z^1(\theta), dW(\theta) \rangle$$

and (Y^2, Z^2) satisfy

$$Y^2(t) = \Upsilon^2 + \int_t^T M(\theta)d\theta - \int_t^T \langle Z^2(\theta), dW(\theta) \rangle,$$

where $M(\theta)$ is a scalar progressively \mathcal{F}_θ -measurable process. Suppose that $\Upsilon^1 \leq \Upsilon^2$ and $\tilde{G}^1(t, Y^2(t), Z^2(t)) \leq M(t)$. Then we can choose

$$\tilde{G}^2(t, \kappa(t), Y^2, Z^2) = \tilde{G}^1(t, \kappa(t), Y^2, Z^2) + [M(t) - G^1(t, \kappa(t), Y^2(t), Z^2(t))]$$

and apply the result of theorem 3 to deduce that $Y^1(t) \leq Y^2(t)$. If in addition $\tilde{G}^1(t, \kappa(t), Y^2, Z^2) < M(t)$ on a set of positive measure $dt \times dP$, then $Y^1(s) < Y^2(s)$.

4 Viscosity solution to nonlinear parabolic system

In this section we show that a solution of a forward-backward stochastic differential equation generates a viscosity solution of the Cauchy problem for a system of quasilinear parabolic equations.

Let $(\xi(t) \in R^d, y(t) \in R^{d_1}, z(t) \in R^{d \times d_1})$ be a solution of the FBSDE

$$d\xi(t) = a(\xi(t))dt + A(\xi(t))dw(t), \quad \xi(s) = x, \quad (4.1)$$

$$dy(t) = -\Gamma^*(t)g([\Gamma^*]^{-1}(t)y(t), [\Gamma^*]^{-1}(t)z(t))dt + z(t)dw(t), \quad (4.2)$$

$$y(T) = \Gamma^*(s, T)u_0(\xi(T)),$$

where $\Gamma(t)$ is a multiplicative operator functional of the process $\xi(t)$ generated by the solution $\eta(t) \in R^{d_1}$ of the linear SDE

$$d\eta(t) = c(\xi(t))\eta(t)dt + C(\xi(t))(\eta(t), dw(t)), \quad \eta(s) = h, \quad (4.3)$$

and $u_0 : R^d \rightarrow R^{d_1}$ be a continuous bounded function.

Denote by $S_+^{d_1} = \{h \in R^{d_1} : h_m \geq 0, m = 1, \dots, d_1 \text{ and } \|h\| = 1\}$, and let e_1, \dots, e_{d_1} be a fixed orthonormal basis in R^{d_1} .

In section 2 we have shown that one can write (4.2) in the form

$$dy(t) = -f(t, \xi(t), y(t), z(t))dt + z(t)dw(t), \quad y(T) = \Gamma^*(s, T)u_0(\xi(T)), \quad (4.4)$$

and proved that given a solution $\xi(t)$ of (4.1), there exists a unique solution $(y(t), z(t))$ of this BSDE.

Assume that there exists a solution $(\xi_{s,x}(t), y^{s,x}(t), z^{s,x}(t))$ to (4.1), (4.2) and the comparison theorem 2 is valid. The aim of this section is to prove that the function $u(s, x) = y^{s,x}(s)$ is a viscosity solution of the Cauchy problem

$$\begin{aligned} \frac{\partial u_l}{\partial s} + \frac{1}{2}Tr A^*(x)\nabla^2 u_l A(x) + \langle a(x), \nabla u_l \rangle + \\ + B_{lm}^i(x)\nabla_i u_m + c_{lm}(x)u_m + g_l(x, u, K(u, \nabla u)) = 0, \quad l = 1, \dots, d_1, \\ u(T, x) = u_0(x), \end{aligned} \quad (4.5)$$

where $B_{lm}^i = \sum_{q=1}^d C_{lm}^q A^{q_i}$, $K(u, \nabla u) = C^*u + A^*\nabla u$.

As it was mentioned in section 2 the system (4.5) can be easily reduced to a scalar parabolic equation

$$\frac{\partial V}{\partial s} + \frac{1}{2}Tr Q^*(x, h)\nabla^2 V Q(x, h) + \langle q(x, h), \nabla V \rangle + G(h, x, V, Q^*\nabla V) = 0, \quad (4.6)$$

$$V(T, x) = V_0(x, h) = \langle h, u_0(x) \rangle$$

with respect to a scalar function V defined on $[0, T] \times R^d \times S_+^{d_1}$ (see equation (2.1)).

Hence we recall first the definition of a viscosity solution of the Cauchy problem for a general scalar nonlinear parabolic equation

$$\frac{\partial V}{\partial s} + \Psi(s, z, V, \nabla V, \nabla^2 V) = 0. \quad V(T, z) = V_0(z), \quad (4.7)$$

where $z = (x, h)$.

A function $\Psi : [0, T] \times (R^d \times S_+^{d_1}) \times R \times (R^d \times R^{d_1}) \times R^d \otimes R^d \rightarrow R$ satisfying estimates

$$\Psi(s, z, V, p, q) \leq \Psi(s, z, U, p, q) \quad \text{if } V \leq U,$$

and

$$\Psi(s, z, V, p, q) \leq \Psi(s, z, V, p, q_1) \quad \text{if } q_1 \leq q$$

is called a proper function.

Given a proper function Ψ to define a viscosity solution of (4.7) one has to introduce notions of a sub- and a supersolution of this Cauchy problem.

Denote by $C_{d,d_1}^{1,2} \equiv C^{1,2}([0, T] \times R^d; R^{d_1})$ a set of functions $\psi : [0, T] \times R^d; R^{d_1}$ differentiable in $s \in [0, T]$ and twice differentiable in $x \in R^d$.

A continuous real valued function $V(s, z)$ is called a subsolution of (4.7) if $V(T, z) \leq V_0(z)$, $z \in R^{d_2}$, $d_2 = d + d_1$, and for any $\Phi \in C_{d_2,1}^{1,2}$ and a point $(s, z) \in [0, T] \times R^{d_2}$ which is a local maximum of $V(t, \tilde{z}) - \Phi(t, \tilde{z})$ the inequality

$$\frac{\partial \Phi}{\partial s} + \Psi(s, z, V, \nabla \Phi, \nabla^2 \Phi) \geq 0$$

holds.

A continuous function $V(s, z)$ is called a super-solution of (4.7) if $V(T, z) \geq V_0(z)$, $z \in R^{d_2}$ and for any $\phi \in C_{d_2,1}^{1,2}$ and $(s, x) \in [0, T] \times R^d$ which is a local minimum of $u_m(t, \tilde{x}) - \phi_m(t, \tilde{x})$ the inequality

$$\frac{\partial \Phi}{\partial s} + \Psi(s, z, V, \nabla \Phi, \nabla^2 \Phi) \leq 0$$

holds. A continuous function $V(s, z)$ is called a viscosity solution of (4.7), if it is both sub- and super-solution of this Cauchy problem. Hence to prove that the function $V(s, z)$ is a viscosity solution to (4.7) one has to prove that V is both sub- and supersolution of (4.7).

To give a definition of a viscosity solution of the Cauchy problem to the system (4.5) we use a definition of a viscosity solution of the scalar Cauchy problem (4.6) and then rewrite the definition in terms of the solution to (4.5).

Given functions $\phi_m \in C_{d,d_1}^{1,2}$, $m = 1, \dots, d_1$ denote by

$$[\mathcal{A}\phi]_m(x) = \frac{1}{2} \text{Tr} A^*(x) \nabla^2 \phi_m A(x) + \langle a(x), \nabla \phi_m \rangle + B_{ml}^i(x) \nabla_i \phi_l + c_{ml}(x) \phi_l,$$

where $i = 1, \dots, d$, $m, l = 1, \dots, d_1$.

Let $(s, x, \phi, p, q) \in [0, T] \times R^d \times R^{d_1} \times R^{d \times d_1} \times R^{d^2 \times d_1}$ and

$$\begin{aligned} \mathcal{M}_m(s, x, \phi, p, q_m) &= \frac{1}{2} \text{Tr} A^*(x) q_m A(x) + \langle a(x), p_m \rangle + \\ &+ B_{ml}^i(x) \nabla_i p_l + c_{ml}(x) \phi_l + g_l(s, x, u, p). \end{aligned} \quad (4.8)$$

Given \mathcal{M}_m , $m = 1, \dots, d_1$, of the form (4.8) the system

$$\frac{\partial u_m}{\partial s} + \mathcal{M}_m(s, x, u, \nabla u, \nabla^2 u_m) = 0 \quad (4.9)$$

coincides with (4.5).

A continuous function $u : [0, T] \times R^d \rightarrow R^{d_1}$ is called a sub-solution of (4.9) if for each $m = 1, \dots, d_1$ an inequality $u_m(T, x) \leq u_{0m}(x)$, holds and for any $\varphi_m \in C_{d,1}^{1,2}$ and a point $(s, x) \in [0, T] \times R^d$ which is a local maximum of $u_m(\tilde{s}, \tilde{x}) - \varphi_m(\tilde{s}, \tilde{x})$ an inequality

$$\frac{\partial \varphi_m}{\partial s} + \mathcal{M}(s, x, u, \nabla \varphi, \nabla^2 \varphi_m) \geq 0 \quad (4.10)$$

holds.

A continuous function $u(s, x)$ is called a super-solution of (4.9) if for each $m = 1, \dots, d_1$ an inequality $u_m(T, \tilde{x}) \geq u_{0m}(\tilde{x})$, $x \in R^d$ holds and for any $\varphi_m \in C_{d,1}^{1,2}$ and a point $(s, x) \in [0, T] \times R^d$ which is a local minimum of $u_m(\tilde{s}, \tilde{x}) - \varphi_m(\tilde{s}, \tilde{x})$ an inequality

$$\frac{\partial \varphi_m}{\partial \tilde{s}} + \mathcal{M}_m(s, x, u, \nabla \varphi, \nabla^2 \varphi_m) \leq 0, \quad (4.11)$$

holds.

A continuous function $u(s, x)$ is called a viscosity solution of (4.9), if it is both sub- and super-solution of this Cauchy problem. Hence to prove that the function $u(s, x)$ is a viscosity solution to (4.9) one has to prove that u is both sub- and super-solution of (4.9).

Theorem 4.1. Assume that conditions of theorem 2 hold and $(\xi_{s,x}(t), y^{s,x}(t), z^{s,x}(t), \eta^{s,x}(t))$ is a solution to (4.1)-(4.3). Then $u(s, x) = y^{s,x}(s)$ is a continuous in (s, x) viscosity solution of (4.5).

Proof. Under assumptions of section 2 continuity of $u(s, x) = y^{s,x}(s)$ in spatial variable x and time variable s is granted by the BSDE theory results [5] which state that under **C 2.1** and **C 2.2** the solution of BSDE (4.4) is continuous with respect to parameters (s, x) . To verify that $u(s, x)$ is a viscosity solution of (4.5), we have to prove that u is both a subsolution and a supersolution of (4.5). First we check that u is a subsolution. To this end for each $m = 1, \dots, d_1$ we can choose a function $\phi_m \in C_{d,1}^{1,2}$ and a point $(s, x) \in [0, T] \times R^d$ such that at the point (s, x) a function $u_m(s, x) - \phi_m(s, x)$ has a local maximum. Without loss of generality we assume that $u_m(s, x) = \phi_m(s, x)$.

We have to prove that (4.10) holds.

Assume on the contrary that there exists $m \in \{1, \dots, d_1\}$ such that

$$\mathcal{K}_m^{u,\phi}(s, x) = \frac{\partial \phi_m}{\partial s} + [\mathcal{A}\phi]_m(s, x) + g_m(s, x, u(s, x), K(u, \nabla \phi)(s, x)) < 0. \quad (4.12)$$

By continuity there exists $0 < \alpha \leq T - s$ such that for all $\theta \in [s, s + \alpha]$, $x_1 \in R^d$, $h_1 \in R^{d_1}$, $\|x - x_1\| \leq \alpha$, $\|e_m - h_1\| \leq \alpha$ the inequalities

$$\Phi^u(\theta, x_1, h_1) - \Phi^\phi(\theta, x_1, h_1) \leq 0 \quad (4.13)$$

and

$$\langle h_1, \left(\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi \right)(\theta, x_1) + g(\theta, x_1, u(\theta, x_1), K(u, \nabla \phi)(\theta, x_1)) \rangle < 0 \quad (4.14)$$

hold.

Given $(\xi_{s,x}(t), \eta_{s,h}(t))$ satisfying (4.1), (4.3), define τ by

$$\tau = \inf\{t \geq s : \|\xi_{s,x}(t) - x\| \geq \alpha\} \wedge \inf\{t \geq s : \|\eta_{s,h}(t) - h\| \geq \alpha\} \wedge (s + \alpha).$$

It follows from results in [10],[11] that the pair

$$(\hat{y}(t), \hat{z}(t)) = (y^{s,x}(t \wedge \tau), I_{[s,\tau]}(t)z^{s,x}(t \wedge \tau)), \quad s \leq t \leq s + \alpha$$

satisfies BSDE

$$\begin{aligned} \hat{y}(t) &= \Gamma^*(t, \tau)u([s + \alpha] \wedge \tau, \xi([s + \alpha] \wedge \tau)) + \int_t^{s+\alpha} I_{[s,\tau]}(\theta)f(\theta, \xi(\theta), \hat{y}(\theta), \hat{z}(\theta))d\theta - \\ &\quad \int_t^{s+\alpha} \hat{z}(\theta)dw(\theta), \quad s \leq t \leq s + \alpha. \end{aligned} \quad (4.15)$$

On the other hand applying Ito's formula we obtain that the couple

$$(\tilde{y}(t), \tilde{z}(t)) = (\Gamma^*(t, t \wedge \tau)\phi(t \wedge \tau, \xi_{s,x}(t \wedge \tau)), I_{[s,\tau]}(t)K(u, \nabla \phi)(t, \xi_{s,x}(t))), \quad s \leq t \leq s + \alpha,$$

where

$$K(u, \nabla \phi)(t, \xi_{s,x}(t)) = \begin{pmatrix} \Gamma^*(t)A^*(\xi(t))\nabla \phi(t, \xi_{s,x}(t)) \\ \Gamma^*(t)C^*(\xi_{s,x}(t))u(t, \xi_{s,x}(t)) \end{pmatrix}, \quad s \leq t \leq s + \alpha,$$

satisfies a BSDE

$$\begin{aligned} \tilde{y}(t) &= (\Gamma^*(\tau)\phi(\tau, \xi_{s,x}(\tau)) + \int_t^{s+\alpha} I_{[s,\tau]}(\theta) \left(\frac{\partial \phi}{\partial \theta} + [\mathcal{A}\phi] \right)(\theta, \xi_{s,x}(\theta))d\theta + \\ &\quad \int_t^{s+\alpha} \tilde{z}(\theta)dw(\theta). \end{aligned}$$

Notice that $\hat{y}_m(s) = \tilde{y}_m(s) = u_m(s, x)$.

Then for any stopping time $\tau \in [s, s + \alpha]$ due to (4.13) and (4.14) we derive

$$\begin{aligned} 0 &\geq [\Phi^u(\tau, \kappa(\tau)) - \Phi^\phi(\tau, \kappa(\tau))] = \langle e_m, u(s, x) - \phi(s, x) \rangle - \\ &\quad - \int_s^\tau \langle e_m, [\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi](\theta, \xi_{s,x}(\theta)) \rangle d\theta - \int_s^\tau \langle e_m, f(\theta, \xi_{s,x}(\theta), \hat{y}(\theta), \hat{z}(\theta)) \rangle d\theta + \\ &\quad + \int_s^\tau \langle e_m, [\hat{z}(\theta) - K(u, \nabla \phi)(\theta, \xi_{s,x}(\theta))] \rangle dw(\theta). \end{aligned}$$

Keeping in mind that by assumption for each $m = 1, \dots, d_1$ at the point (s, x) we have $u_m(s, x) - \phi_m(s, x) = 0$ and computing the expectation of both parts of the last inequality we deduce

$$E \left(\int_s^\tau \langle e_m, [\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi](\theta, \xi(\theta)) \rangle d\theta + \int_s^\tau \langle e_m, f(\theta, \xi(\theta), \hat{y}(\theta), \hat{z}(\theta)) \rangle d\theta \right) \geq 0. \quad (4.16)$$

Denote by

$$\begin{aligned} \gamma_1(s, \tau) &= \langle e_m, \int_s^\tau \{ [\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi](\theta, \xi(\theta)) + g(\theta, x_1, u(\theta, \xi(\theta)), K(u, \nabla \phi)(\theta, \xi(\theta))) \} \\ \gamma_2(s, \tau) &= \langle e_m, \int_s^\tau [f(\theta, \xi(\theta), \hat{y}(\theta), \hat{z}(\theta)) - g(\theta, \xi(\theta), u(\theta, \xi(\theta)), K(u, \nabla \phi)(\theta, \xi(\theta)))] d\theta \rangle \end{aligned}$$

and by

$$\gamma_3(s, \tau) = \langle e_m, \int_s^\tau \{f(\theta, \xi(\theta), \hat{y}(\theta), \hat{z}(\theta)) - f(\theta, \xi(\theta), \tilde{y}(\theta), \tilde{z}(\theta))\} d\theta \rangle$$

and rewrite (4.16) in the form

$$E[\gamma_1(s, \tau) + \gamma_2(s, \tau) + \gamma_3(s, \tau)] \geq 0.$$

Assume that there exists a number $\delta_0 < 0$ such that $\mathcal{K}^{u, \phi}(s, x) < \delta_0$ and

$$\tau_1 = \inf\{\theta \in [s, s + \alpha] : \mathcal{K}^{y(\theta), z(\theta)}(\theta, \xi(\theta)) \leq \delta_0\} \wedge \tau.$$

By assumption (4.16) holds for τ and hence for τ_1 . But this leads to a contradiction since

$$0 > \delta_0 E(\tau_1 - s) \geq E \left[\int_s^{\tau_1} \mathcal{N}^{y(\theta), z(\theta)}(\theta, \xi(\theta)) d\theta \right] \geq 0.$$

It remains to check that $\gamma_2(s, s + \Delta s) \rightarrow 0$ and $\gamma_3(s, s + \Delta s) \rightarrow 0$ as $\Delta s \rightarrow 0$ a.s.

Note that $\gamma_2(s, s + \Delta s) \rightarrow 0$ a.s. by definition of f , properties of $\Gamma(s, t)$ and uniqueness of a BSDE solution.

Finally we check that $\gamma_3(s, s + \Delta s) \rightarrow 0$ as $\Delta s \rightarrow 0$ a.s. Note that the couple $(\tilde{y}(t), \tilde{z}(t)), s \leq t \leq s + \Delta s$ satisfies

$$\begin{aligned} \tilde{y}(t) = & \Gamma^*(s + \Delta s) \phi(s + \Delta s, \xi_{s,x}(s + \Delta s)) + \int_t^{s + \Delta s} f(\theta, \xi_{s,x}(\theta), \tilde{y}(\theta), \tilde{z}(\theta)) d\theta - \\ & - \int_t^{s + \Delta s} \tilde{z}(\theta) dw(\theta). \end{aligned} \quad (4.17)$$

Given $\theta \in [s, s + \Delta s]$,

$$\mathcal{K}^{u, \phi}(\theta, x) = \left(\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi \right)(\theta, x) + g(\theta, x, \phi(\theta, x), K(u, \nabla \phi)(\theta, x)),$$

set

$$v(\theta) = \tilde{y}(s + \Delta s) - \Gamma^*(s, \theta) \phi(\theta, \xi_{s,x}(\theta)) - \int_\theta^{s + \Delta s} \mathcal{K}^{u, \phi}(\vartheta, \xi_{s,x}(\vartheta)) d\vartheta$$

and

$$\varpi(\theta) = \tilde{z}(\theta) - K(u, \nabla \phi)(\theta, \xi_{s,x}(\theta)).$$

Applying Ito's formula we derive BSDE to govern the couple $(v(\theta), \varpi(\theta))$

$$\begin{aligned} v(\theta) = & \Gamma^*(s, s + \Delta s) \phi(s + \Delta s, \xi_{s,x}(s + \Delta s)) - \Gamma^*(s, \theta) \phi(\theta, \xi_{s,x}(\theta)) + \\ & + \int_\theta^{s + \Delta s} f(\vartheta, \xi_{s,x}(\vartheta), \tilde{y}(\vartheta), \tilde{z}(\vartheta)) d\vartheta - \int_\theta^{s + \Delta s} \mathcal{K}^{u, \phi}(\vartheta, \xi_{s,x}(\vartheta)) d\vartheta - \\ & - \int_\theta^{s + \Delta s} \tilde{z}(\vartheta) dw(\vartheta) + \int_\theta^{s + \Delta s} K(u, \nabla \phi)(\vartheta, \xi_{s,x}(\vartheta)) dw(\vartheta) = \\ & \int_\theta^{s + \Delta s} f(\vartheta, \xi_{s,x}(\vartheta), v(\vartheta) + \Gamma^*(s, \vartheta) \phi(\vartheta, \xi_{s,x}(\vartheta)) + \\ & + \int_\vartheta^{s + \Delta s} \mathcal{K}^{u, \phi}(r, \xi_{s,x}(r)) dr, \varpi(\vartheta) + K(u, \nabla \phi)(\vartheta, \xi_{s,x}(\vartheta)) d\vartheta + \\ & + \int_\theta^{s + \Delta s} \left[\left(\frac{\partial \phi}{\partial \vartheta} + \mathcal{A}\phi \right)(\vartheta, \xi_{s,x}(\vartheta)) - \mathcal{K}^{u, \phi}(\vartheta, \xi_{s,x}(\vartheta)) \right] d\vartheta - \int_\theta^{s + \Delta s} \varpi(\vartheta) dw(\vartheta). \end{aligned} \quad (4.18)$$

We verify that (v, ϖ) converges to $(0, 0)$ as $\Delta s \rightarrow 0$. Keeping in mind the estimates for the generator g by standard reasoning based on the Ito formula and the Burkholder inequality we can prove that

$$E \left[\sup_{t \in [s, s + \Delta s]} |v(t)|^2 \right] + E \left[\int_s^{s + \Delta s} \|\varpi(\theta)\|^2 d\theta \right] \leq LE \left[\int_s^{s + \Delta s} \|m(\theta, \Delta s)\|^2 d\theta \right],$$

where

$$\begin{aligned} m(\theta, \Delta s) = & -\mathcal{K}^{u, \phi}(\theta, \xi_{s,x}(\theta)) + \left(\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi \right)(\theta, \xi_{s,x}(\theta)) + \\ & f(\theta, \xi_{s,x}(\theta), v(\theta) + \Gamma^*(s, \theta) \phi(\theta, \xi_{s,x}(\theta)) + \\ & + \int_\theta^{s + \Delta s} \mathcal{K}^{u, \phi}(r, \xi_{s,x}(r)) dr, \varpi(\theta) + K(u, \nabla \phi)(\theta, \xi_{s,x}(\theta))). \end{aligned}$$

Furthermore, since $\sup_{\theta \in [s, s+\Delta s]} E[\|\xi_{s,x}(\theta) - x\|^2] \rightarrow 0$ as $\Delta s \rightarrow 0$ and parameters of stochastic equations as well as the function ϕ and its derivatives are uniformly continuous in x , we obtain

$$\lim_{\Delta s \rightarrow 0} \sup_{s \leq \theta \leq s+\Delta s} E[\|m(\theta, \Delta s)\|^2] = 0.$$

Hence,

$$\begin{aligned} E \left[\sup_{s \leq \theta \leq s+\Delta s} |v(\theta)|^2 \right] + E \left[\int_s^{s+\Delta s} \|\varpi(\theta)\|^2 d\theta \right] &\leq \\ LE \left[\int_s^{s+\Delta s} \|m(\theta, \Delta t)\|^2 d\theta \right] &\leq \varepsilon(\Delta s) \Delta s, \end{aligned} \quad (4.19)$$

where $\varepsilon(\Delta s) \rightarrow 0$ as $\Delta s \rightarrow 0$. As a result we get that $\tilde{y}(\theta)$ converges to $\phi(s, x)$ and $\tilde{z}(\theta)$ converges to $[Cu](s, x) + [\nabla \phi A](s, x)$ a.s. as $\Delta s \rightarrow 0$.

This estimate does not satisfy yet our purposes. To get a more satisfactory estimate we evaluate the conditional expectation of both sides of (4.18), that leads to $v(\theta) = E \left[\int_\theta^{s+\Delta s} n(\vartheta, \Delta s) d\vartheta | \mathcal{F}_\theta \right]$, where

$$\begin{aligned} n(\theta, \Delta s) &= -\mathcal{K}^{u, \phi}(\theta, \xi_{s,x}(\theta)) + \left[\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi \right](\theta, \xi_{s,x}(\theta)) + f(\theta, \xi_{s,x}(\theta), \tilde{y}(\theta), \tilde{z}(\theta)) = \\ &= f(\theta, \xi_{s,x}(\theta), \tilde{y}(\theta), \tilde{z}(\theta)) - \\ &- f \left(\theta, \xi_{s,x}(\theta), \Gamma^*(\theta) \phi(\theta, \xi_{s,x}(\theta)) + \int_\theta^{s+\Delta s} \mathcal{K}^{u, \phi}(\theta, \xi_{s,x}(\theta)) d\theta, K(u, \nabla \phi)(\theta, \xi_{s,x}(\theta)) \right). \end{aligned}$$

By Lipschitz continuity of f we have for $s \leq \theta \leq s + \Delta s$, $\|n(\theta, \Delta s)\| \leq L[\|v(\theta)\| + \|\varpi(\theta)\|]$, that is $\|n(\theta, \Delta s)\| \rightarrow 0$ a.s. as $\Delta s \rightarrow 0$.

Hence we have proved that $u(s, x)$ is a viscosity subsolution of the Cauchy problem (4.5). In a similar way we prove that $u(s, x)$ is a supersolution of (4.5) and hence a viscosity solution of this problem.

Acknowledgement Financial support of grant RFBR 12-01-00427-a and the Minobrnauki project 1.370.2011 is gratefully acknowledged

References

- [1] Belopolskaya Ya., Dalecky Yu.: Investigation of the Cauchy problem for systems of quasilinear equations via Markov processes. *Izv. VUZ Matematika*. N 12, 6–17 (1978)
- [2] Belopolskaya Ya., Dalecky Yu.L.: Stochastic equations and differential geometry. Kluwer, Boston, (1990)
- [3] Belopolskaya Ya., Dalecky Yu.: Markov processes associated with nonlinear parabolic systems. *DAN SSSR* **250**, N 3, 268–271 (1980)
- [4] Pardoux E., Peng S.: Adapted solutions of backward stochastic equations. *Systems Control Lett.* **14**, 55–61 (1990)
- [5] Pardoux E., Peng S. : Backward stochastic differential equations and quasilinear parabolic partial differential equations. *Lecture Notes in CIS*, **176**, pp. 200–217. Springer (1992).
- [6] Peng S. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics* **37**, 61–74 (1991)
- [7] Ma J., Yong J.: Forward – Backward Stochastic Differential Equations and Their Applications, *Lecture Notes in Math.*, **1702**, Springer (1999)
- [8] Albeverio S., Belopolskaya Ya.: Probabilistic Approach to Systems of Nonlinear PDEs and Vanishing Viscosity Method Markov Processes and Related Fields **12**, 1, 59–94 (2006)
- [9] Crandall M., Ishii H., Lions P.: User’s guide to viscosity solutions of second order partial differential equations. *Bull. AMS* **27**, 1, 1–67 (1992)
- [10] Pardoux E. : Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. *Stochastic Analysis and Related Topics: The Geilo Workshop*, 79–127, Birkhäuser (1996)
- [11] Pardoux E., Tang S.: Forward – backward stochastic differential equations and quasilinear parabolic PDEs. *Probab. Theory Related Fields* **114**, 2, 123–150 (1999)